*J. Fluid Mech.* (2006), *vol.* 550, *pp.* 139–148.  
doi:10.1017/S002211200500830X Printed in the United Kingdom

# Self-propulsion in a viscous fluid: arbitrary surface deformations

By EHUD YARIV

Faculty of Mechanical Engineering, Technion – Israel Institute of Technology, Haifa 32000, Israel

(Received 12 September 2005 and in revised form 27 November 2005)

The self-propulsion of a generally deformable body at low-Reynolds-number conditions is discussed. The translational and rotational velocities of the body relative to an inertial reference system are presented as surface quadratures using a Lagrangian 'body-fixed' shape description. The power dissipated into the fluid is obtained as a quadratic functional of the surface deformation rate. For symmetric strokes, the net displacement obtained by the execution of a single deformation cycle is provided by a functional of the intrinsic swimmer shape and its time derivative.

## 1. Introduction

In the absence of external force fields, energetic reasoning necessitates that any rigid body would eventually come to rest when placed in a quiescent viscous fluid. Such reasoning does not apply for a non-rigid body, which can transfer energy into the fluid by changing its shape. Such a body ('swimmer') may persistently propel itself relative to the carrying fluid by executing periodic shape deformations using an internal (e.g. electrical or chemical) energy source. Usually, this motion is achieved by intentional time-periodic shape deformations of the swimmer ('strokes').

When the swimmer is sufficiently small, the fluid motion is governed by the linear Stokes equations. Self-propulsion in the Stokes regime has received significant attention (Lighthill 1975; Brennen & Winet 1977; Purcell 1977; Childress 1981; Stone & Samuel 1996; Magar & Pedley 2005) owing to its relevance to the motion of micro-organisms. Today, with the rapid advance in miniaturization technology, it has attracted renewed interest (Becker, Koehler & Stone 2003; Avron, Gat & Kenneth 2004). Given the reversibility of the creeping-flow equations, various properties of 'Stokes swimmers' may be deduced without the need to explicitly solve the governing equations (Purcell 1977). Perhaps the most familiar of these is the 'Scallop theorem', stating that a reversible stroke cannot yield any net displacement. A more general observation is that swimming in the Stokes regime is described by a purely geometric process: the net displacement of a swimmer during a stroke is independent of the rate at which it was executed.

Despite the generality of the above statements, it is still quite challenging to analyse the locomotion of Stokes swimmers. One encounters difficulties just by trying to formulate a well-posed self-propulsion problem for the case of arbitrary (that is, non-infinitesimal) surface deformations. For example, it is not *a priori* obvious how to translate the intuitive notions of 'shape' and 'displacement' into a convenient mathematical description that would naturally represent the time-periodic body deformation and allow a tractable analysis of the fluid-mechanical problem. Another challenge stems from the conflict between two different formulations of the coupled swimmer–fluid system: while the flow field is most conveniently analysed using an Eulerian description, the requisite link between the velocity field and the evolving body shape requires a Lagrangian description. It is also unclear how to merge the 'body-fixed' shape description with the eventual swimmer's 'rigid-body' motion relative to an inertial reference system.

Several of these issues were discussed by Shapere & Wilczek (1989), where the locomotion problem was formulated using a gauge-field description. With this method, the net displacement and re-orientation generated by a deformation cycle are provided by a line integral of a gauge potential, which may in principle be calculated by solving a boundary-value problem. While this approach is illuminating in addressing the above-mentioned fundamental concepts, it does not provide operational formulae for the locomotion resulting from arbitrary shape deformations. Indeed, the three-dimensional examples provided in Shapere & Wilczek (1989) address small deformations from cylinders and sphere, for which the transformation from the gauge-field formulation to a well-posed hydrodynamic problem is straightforward; for such shapes, however, the gauge-field approach does not seem superior to more conventional techniques (see e.g. Stone & Samuel 1996). In fact, most of the specific systems discussed in the context of self-propulsion in the Stokes regime involve small deviations from simple shapes, such as planes (Taylor 1951), cylinders (Blake 1971), and spheres (Lighthill 1952; Stone & Samuel 1996; Magar & Pedley 2005). For such geometries, the flow field is calculated analytically using regular perturbation expansions. In these domain-perturbation analyses, the conceptual issues of proper Lagrangian definition of a deforming shape become degenerate and may usually be ignored.

In this paper I address these conceptual issues. A geometrically well-posed problem is formulated using two reference systems: a laboratory system, relative to which the swimmer propagation is described, and a 'body-fixed' system ('attached' to the swimmer) whose translational and rotational motions are not known *a priori*. Once the linkage between the two systems is clarified, the hydrodynamics is resolved by a straightforward use of the Lorentz reciprocal theorem. This procedure provides a set of differential equations that describes the 'external' motion of a swimmer as a function of its 'intrinsic' deformation pattern. This generic approach can illuminate several properties of Stokes swimming in a broader context than that obtained by the solution of specific configurations. The focus on arbitrary body deformation necessitates the translation of intuitive notions such as 'body shape' and 'stroke' into a convenient well-defined mathematical structure, which should appeal to the fluid-mechanics community. In a sense, I attempt here to suggest a comparable model to that obtained by Miloh & Galper (1993) for locomotion in a perfect fluid (see also Miloh 1991). Indeed, several of the concepts employed by Miloh & Galper (1993) are found useful in the present analysis. However, the presence of the no-slip condition in a viscous fluid, which necessitates a Lagrangian description in the delineation of the swimmer's surface deformation, implies a qualitative difference which emerges in the problem formulation stage, even prior to the hydrodynamic analysis: a Lagrangian formalism is obviously redundant in the analysis of perfect-fluid locomotion, where the fluid is affected only by velocities which are normal to the surface.

## 2. Geometry

We address the locomotion of a homogeneous swimmer which deforms its shape in a prescribed periodic manner. Conceptually, this cyclic motion is described in a

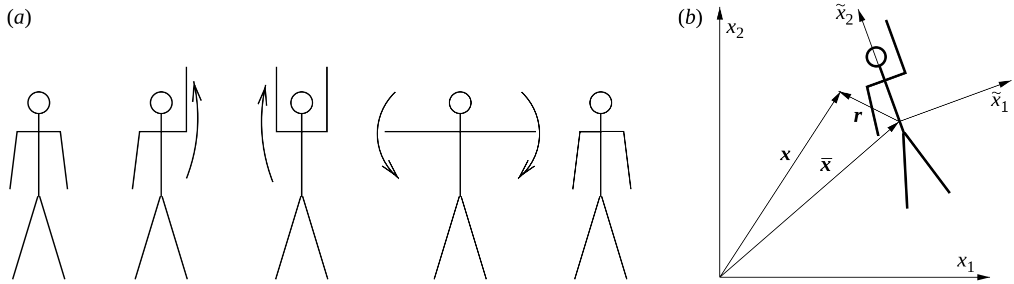


FIGURE 1. ($a$) A schematic body-fixed description of an irreversible and asymmetric deformation cycle, from $t = 0$ (left) to $t = T$ (right). This sequence represents the swimmer's shape as seen by an observer positioned at $\tilde{S}$. ($b$) Schematic description of the inertial and body-fixed coordinate systems.

'body-fixed' reference frame $\tilde{S} = \tilde{x}_1\tilde{x}_2\tilde{x}_3$, in which the swimmer returns to its original shape after completing a single deformation cycle during a time interval of length $T$ (which with no loss of generality is assumed to begin at $t = 0$). A schematic of a such a cycle appears in figure 1 ($a$). Owing to the non-rigidity of the body, the term 'body-fixed' is not uniquely defined: a given periodic deformation may be described via many reference frames. Nevertheless, for our purpose any reference frame relative to which the body returns to its original shape is sufficient. With no loss of generality, the origin of the body-fixed coordinate system is assumed to coincide with the swimmer's centroid at all times.

When the swimmer is placed in an otherwise quiescent fluid domain, self-propulsion occurs. It is convenient to describe the swimmer motion in an inertial reference frame $S = x_1x_2x_3$ attached to the far-field quiescent fluid. The (time-dependent) domain possessed by the body is denoted by $D(t)$; the corresponding enclosed volume, surrounding boundary, and outward unit normal are respectively denoted by $V(t)$, $\partial D(t)$, and $\boldsymbol{n}$. Conceptually, the 'rigid-body' motion relative to $S$ consists of the time evolution of the swimmer's position (that is, its centroid position) and orientation (that is, the orientation of $\tilde{S}$) relative to $S$. The centroid $\bar{\boldsymbol{x}}$ is defined by the relation

$$V\bar{\boldsymbol{x}} = \int_D \mathrm{d}V\,\boldsymbol{x}, \tag{2.1}$$

with $\mathrm{d}V$ being a differential volume element and $\boldsymbol{x}$ a position vector measured from the origin of $S$.

To formulate a tractable hydrodynamic problem, the boundary $\partial D$ must be defined in a convenient matter. This is done by uniquely identifying each point on $\partial D$ with two parameters, say $s_1$ and $s_2$. Since it is desired to relate the time evolution of the body surface to the adjacent fluid velocity, it is natural to employ a material parameterization. Thus, the entity $(s_1, s_2)$ is chosen to represent a single material point located on $\partial D$. The position $\boldsymbol{x} = (x_1, x_2, x_3)$ of this point relative to the inertial frame $S$ generally evolves with time, say $\boldsymbol{x} = \boldsymbol{F}(s_1, s_2; t)$. Of course, material points $\partial D$ do not return (in general) to their original position in $S$ following the completion of a swimming cycle. How then does the above-suggested Lagrangian description relate to the prescribed periodic deformation in the body-fixed system?

In the terminology of Shapere & Wilczek (1989), the function $\boldsymbol{F}$ describes a 'located' shape, whereas the periodic deformation describes an 'unlocated' shape. An unlocated shape constitutes an equivalence class of all geometrically identical located shapes, regardless of their position and orientation relative to $S$. (In the absence of surrounding fluid, the located and unlocated shapes 'coincide.') The requisite link between the two descriptions relies upon a relative surface description, using the swimmer's centroid as a reference point. This description is provided by the system $\tilde{S}$, which, being attached to the centroid, moves at the (yet unknown) time-dependent velocity $\mathrm{d}\bar{\boldsymbol{x}}/\mathrm{d}t$. If the shape deformation is sufficiently symmetric (cf. § 4), the swimmer 'does not rotate', meaning that $\tilde{S}$ does not rotate.† In general, however, $\tilde{S}$ rotates relative to $S$ with an angular velocity $\boldsymbol{\Omega}$, which is conveniently interpreted as the swimmer's angular velocity. Thus, the three unit vectors of $\tilde{S}$, $\tilde{\boldsymbol{e}}_1$, $\tilde{\boldsymbol{e}}_2$, and $\tilde{\boldsymbol{e}}_3$, are time dependent. A schematic of the two coordinate systems is provided in figure 1 (*b*).

It is therefore convenient to define the position vector measured relative to the centroid,

$$\boldsymbol{r} = \boldsymbol{x} - \bar{\boldsymbol{x}}, \tag{2.2}$$

and parameterize $\partial D$ through the collection of the values of $\boldsymbol{r}$ on the boundary, $\boldsymbol{r}|_{\partial D}$. This Lagrangian description is captured by the definition

$$\boldsymbol{f}(s_1, s_2; t) = \boldsymbol{F}(s_1, s_2; t) - \bar{\boldsymbol{x}}. \tag{2.3}$$

The respective $\tilde{S}$-coordinates of this function, $\boldsymbol{f} \cdot \tilde{\boldsymbol{e}}_i$, are denoted by $\tilde{f}_i$. Using this formulation, it is now possible to define the underlying concepts of self-propulsion in a manner consistent with our intuitive notions: a swimming stroke of duration $T$ is defined as the time evolution of $\tilde{f}_1$, $\tilde{f}_2$, and $\tilde{f}_3$ (corresponding to the unlocated shape as seen by an observer in $\tilde{S}$) during the interval $0 < t < T$. (This evolution must satisfy obvious continuity conditions with respect to the arguments $s_1$, $s_2$, and $t$.) In terms of this Lagrangian description, the velocity of the point $(s_1, s_2)$ on $\partial D$ is given by

$$\boldsymbol{v} = \frac{\mathrm{d}\bar{\boldsymbol{x}}}{\mathrm{d}t} + \boldsymbol{\Omega} \times \boldsymbol{f} + \frac{\partial \boldsymbol{f}}{\partial t}, \tag{2.4}$$

where $\partial \boldsymbol{f}/\partial t$ denotes the velocity measured in $\tilde{S}$:

$$\frac{\partial \boldsymbol{f}}{\partial t} = \tilde{\boldsymbol{e}}_i \frac{\partial \tilde{f}_i}{\partial t}. \tag{2.5}$$

(The Einstein summation convention is employed.) In this definition, the partial derivatives are performed for fixed $s_1$ and $s_2$ (that is, for a fixed material point).

Only the unlocated shape (or, more precisely, the components $\tilde{f}_1$, $\tilde{f}_2$, and $\tilde{f}_3$) can be in the direct control of the swimmer during the process of locomotion. (In realistic situations the swimmer may only control 'part' of that shape.) The time evolution of the centroid position $\bar{\boldsymbol{x}}$ and the orientation of $\tilde{S}$ are not directly controlled, but are instead determined by the interaction of the swimmer with the fluid.‡ Both variables continuously modify the swimmer's located configuration, measured relative to the inertial system $S$.

It should be emphasized that not all continuous time-periodic functions are admissible candidates for $\boldsymbol{f}$. Following Miloh (1983), consider the Reynolds transport

† More precisely: it is possible to find $\tilde{S}$ which does not rotate. In principle, $\tilde{S}$ may rotate, but it must return to its original orientation at the end of the stroke.

‡ Owing to the lack of uniqueness in the definition of $\tilde{S}$, $\boldsymbol{\Omega}$ (but not $\mathrm{d}\bar{\boldsymbol{x}}/\mathrm{d}t$) depends upon the specific system $\tilde{S}$ chosen to represent the periodic deformation. However, the net change in the orientation of $\tilde{S}$ relative to $S$ is independent of this choice.

theorem applied to the material within the body:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_D \mathrm{d}V\,\boldsymbol{x} = \oint_{\partial D} \mathrm{d}A\,\boldsymbol{n}\cdot\boldsymbol{v}\boldsymbol{x}. \tag{2.6}$$

The left-hand side of this equation, using (2.1), is

$$\frac{\mathrm{d}V}{\mathrm{d}t}\bar{\boldsymbol{x}} + V\frac{\mathrm{d}\bar{\boldsymbol{x}}}{\mathrm{d}t}, \tag{2.7}$$

whereas, using (2.2), the right-hand side of (2.6) becomes

$$\frac{\mathrm{d}V}{\mathrm{d}t}\bar{\boldsymbol{x}} + \oint_{\partial D} \mathrm{d}A\,\boldsymbol{n}\cdot\boldsymbol{v}\boldsymbol{r}. \tag{2.8}$$

Thus, (2.6) is equivalent to

$$V\frac{\mathrm{d}\bar{\boldsymbol{x}}}{\mathrm{d}t} = \oint_{\partial D} \mathrm{d}A\,\boldsymbol{n}\cdot\boldsymbol{v}\boldsymbol{r}. \tag{2.9}$$

Expansion of the integral using (2.4) yields

$$\frac{\mathrm{d}\bar{\boldsymbol{x}}}{\mathrm{d}t}\cdot\oint_{\partial D} \mathrm{d}A\,\boldsymbol{n}\boldsymbol{r} + \oint_{\partial D} \mathrm{d}A\,\boldsymbol{n}\cdot(\boldsymbol{\Omega}\times\boldsymbol{r})\boldsymbol{r} + \oint_{\partial D} \mathrm{d}A\,\boldsymbol{n}\cdot\frac{\partial \boldsymbol{f}}{\partial t}\boldsymbol{r}. \tag{2.10}$$

The gradient theorem yields $V\mathrm{d}\bar{\boldsymbol{x}}/\mathrm{d}t$ for the first term in the above expression; use of Gauss theorem, in conjunction with the relation $\int_D \mathrm{d}V\,\boldsymbol{r} = \boldsymbol{0}$ (implied by (2.1)–(2.2)), reveals that the second term vanishes. Thus, the following consistency relation emerges:

$$\oint_{\partial D} \mathrm{d}A\,\boldsymbol{n}\cdot\frac{\partial \boldsymbol{f}}{\partial t}\boldsymbol{f} = \boldsymbol{0}. \tag{2.11}$$

This condition implies that the 'control variable' $\boldsymbol{f}$ cannot be any arbitrary function, as would indeed be expected by its geometric construction.

In applications to specific swimmers, the deformation $\boldsymbol{f}$ may also be restricted by additional constraints, reflecting the internal machinery of the swimmer. For example, in the case of an incompressible swimmer the Reynolds transport theorem yields $\oint_{\partial D} \mathrm{d}A\,\boldsymbol{n}\cdot\partial\boldsymbol{f}/\partial t = 0$. Other constraints may represents surface properties (e.g. inextensibility). Note however that such additional constraints are system-specific, and are fundamentally different from the consistency relation (2.11), which is implied by the mere definition of $\boldsymbol{f}$.

## 3. Hydrodynamics

When a shape deformation is performed in a viscous fluid of uniform viscosity $\mu$, it results in a velocity field $\boldsymbol{u}$ and a corresponding stress field $\mu\boldsymbol{\sigma}$. The flow problem consists of (i) the Stokes equations; (ii) the no-slip boundary condition, $\boldsymbol{u} = \boldsymbol{v}$, applied on $\partial D(t)$; (iii) an attenuation condition at infinity; and (iv) the requirement that at any moment the fluid exerts zero force and torque (say, about the centroid) upon the swimmer:

$$\oint_{\partial D} \mathrm{d}A\,\boldsymbol{n}\cdot\boldsymbol{\sigma} = \boldsymbol{0}, \qquad \oint_{\partial D} \mathrm{d}A\,\boldsymbol{r}\times(\boldsymbol{n}\cdot\boldsymbol{\sigma}) = \boldsymbol{0}. \tag{3.1a,b}$$

When seeking a generic description of the swimmer motion, the requisite expressions must eventually be expressed through the unlocated shape, whose description precedes that of located shape. Indeed, when the strokes performed by a swimmer are indicated, one always implicitly describes them in the body-fixed system, attached to the swimmer (see e.g. figure 1 *a*). Accordingly, it is desirable to derive operational equations which depend only upon the $\tilde{S}$-components of $\boldsymbol{f}$ and $\partial \boldsymbol{f}/\partial t$.

Owing to the linearity of the governing equations, the flow field is conveniently decomposed into three separate problems, corresponding to the decomposition of the surface velocity (2.4): in the first problem, the boundary condition is $\boldsymbol{u} = \mathrm{d}\bar{\boldsymbol{x}}/\mathrm{d}t$, in the second it is $\boldsymbol{u} = \boldsymbol{\Omega} \times \boldsymbol{r}$, and in the third $\boldsymbol{u} = \partial \boldsymbol{f}/\partial t$. These conditions apply at the time-evolving shape $\partial D(t)$. In all three problems the flow satisfies the Stokes equations and attenuates at large distances from the swimmer. The flow field in the first problem is identical to that due to a rigid translation of the swimmer with a translational velocity $\mathrm{d}\bar{\boldsymbol{x}}/\mathrm{d}t$, while that of the second problem is identical to that due to a rigid rotation of the swimmer with a rotational velocity $\boldsymbol{\Omega}$. The velocity field in the third problem is generated by a prescribed 'deformational' surface velocity over $\partial D$.

The stress field is accordingly decomposed into three corresponding contributions: translational, linear in $\mathrm{d}\bar{\boldsymbol{x}}/\mathrm{d}t$; rotational, linear in $\boldsymbol{\Omega}$; and deformational, $\boldsymbol{\sigma}_D$, a linear functional of $\partial \boldsymbol{f}/\partial t$:

$$\boldsymbol{\sigma} = \boldsymbol{\Sigma}_T \cdot \frac{\mathrm{d}\bar{\boldsymbol{x}}}{\mathrm{d}t} + \boldsymbol{\Sigma}_R \cdot \boldsymbol{\Omega} + \boldsymbol{\sigma}_D. \tag{3.2}$$

Here, $\boldsymbol{\Sigma}_T = \boldsymbol{\sigma}_{T,i}\tilde{\boldsymbol{e}}_i$ and $\boldsymbol{\Sigma}_R = \boldsymbol{\sigma}_{R,i}\tilde{\boldsymbol{e}}_i$ respectively denote the translational and rotational stress triadics (Happel & Brenner 1965). Explicitly, $\boldsymbol{\sigma}_{T,i}$ denotes the fluid stress (normalized with $\mu$) resulting from a translation of the swimmer with unit velocity along the $\tilde{x}_i$-axis. Similarly, $\boldsymbol{\sigma}_{R,i}$ denotes the fluid stress (normalized with $\mu$) resulting from a rotation of the swimmer with unit velocity about the $\tilde{x}_i$-axis. The triadics $\boldsymbol{\Sigma}_T$ and $\boldsymbol{\Sigma}_R$ are geometric body-fixed functions of the swimmer shape and the fluid position in space. Specifically, they are independent of the fluid viscosity.

Given the equilibrium relations (3.1), the velocities $\mathrm{d}\bar{\boldsymbol{x}}/\mathrm{d}t$ and $\boldsymbol{\Omega}$ are obtained by the solution of a mobility problem engendered by the hydrodynamic force and torque, $\mu\boldsymbol{F}_D$ and $\mu\boldsymbol{T}_D$, which are generated by the deformational flow part. Using a variant of the Lorentz reciprocal theorem (Brenner 1964), these entities can be expressed as the following quadratures (cf. Stone & Samuel 1996):

$$\boldsymbol{F}_D = \oint_{\partial D} \mathrm{d}A\, \boldsymbol{n} \cdot \boldsymbol{\Sigma}'_T \cdot \frac{\partial \boldsymbol{f}}{\partial t}, \quad \boldsymbol{T}_D = \oint_{\partial D} \mathrm{d}A\, \boldsymbol{n} \cdot \boldsymbol{\Sigma}'_R \cdot \frac{\partial \boldsymbol{f}}{\partial t}. \tag{3.3}$$

The prime denotes transposition from the right. The particle velocities are therefore given by the following formulae:

$$\frac{\mathrm{d}\bar{\boldsymbol{x}}}{\mathrm{d}t} = \boldsymbol{M}_T \cdot \boldsymbol{F}_D + \boldsymbol{M}'_C \cdot \boldsymbol{T}_D, \qquad \boldsymbol{\Omega} = \boldsymbol{M}_C \cdot \boldsymbol{F}_D + \boldsymbol{M}_R \cdot \boldsymbol{T}_D. \tag{3.4}$$

Here, $\mu^{-1}\boldsymbol{M}_T$ is the translational mobility dyadic, $\mu^{-1}\boldsymbol{M}_R$ the rotational mobility dyadic, and $\mu^{-1}\boldsymbol{M}_C$ the coupling mobility dyadic (Kim & Karrila 1991). The tensors $\boldsymbol{M}_T$, $\boldsymbol{M}_R$, and $\boldsymbol{M}_C$ are geometric body-fixed functions of the swimmer shape, and do not depend upon its position in $S$ or upon the fluid viscosity.

The term 'body-fixed tensor' means that knowledge of the instantaneous swimmer shape (namely $\tilde{f}_1$, $\tilde{f}_2$, and $\tilde{f}_3$) provides the components of the tensor in $\tilde{S}$: a complete specification requires also the orientation of $\tilde{S}$, namely the unit vectors $\tilde{\boldsymbol{e}}_1$, $\tilde{\boldsymbol{e}}_2$, and $\tilde{\boldsymbol{e}}_3$. This is represented mathematically by the following functional relation:

$$\boldsymbol{M}_T = \boldsymbol{M}_T[\{\tilde{f}_i(t)\};\{\tilde{\boldsymbol{e}}_i(t)\}] \tag{3.5}$$

together with similar expressions for $\boldsymbol{M}_R$ and $\boldsymbol{M}_C$. The notation $\tilde{f}_i(t)$ $(i = 1, 2, 3)$ signifies the complete set of values taken by $\tilde{f}_i$ for all values of $s_1$ and $s_2$ (rather than the specific value at a given material point).† Since the stress triadics $\boldsymbol{\Sigma}_T$ and $\boldsymbol{\Sigma}_R$ also depend upon the position of the fluid point on $\partial D$, they are provided by the functional dependence

$$\boldsymbol{\Sigma}_T = \boldsymbol{\Sigma}_T[\{\tilde{f}_i(t)\}; \{\tilde{\boldsymbol{e}}_i(t)\}; s_1, s_2], \tag{3.6}$$

with a similar expression for $\boldsymbol{\Sigma}_R$.

The formulae (3.3)–(3.4) are still unsatisfactory, since both $\mathrm{d}A$ and $\boldsymbol{n}$ depend upon the evolving swimmer shape. Making use of the definition of a directed area element in terms of the surface parameterization $\boldsymbol{x} = \boldsymbol{F}(s_1, s_2; t)$,

$$\mathrm{d}A\, \boldsymbol{n} = \mathrm{d}s_1\, \mathrm{d}s_2\, \frac{\partial \boldsymbol{F}}{\partial s_1} \times \frac{\partial \boldsymbol{F}}{\partial s_2}, \tag{3.7}$$

and noting (see (2.3)) that $\boldsymbol{F}$ can be replaced by $\boldsymbol{f}$ in the above relation, furnishes the instantaneous specification of the swimmer translational and rotational velocities in terms of the $\tilde{S}$-components of $\boldsymbol{f}$ and $\partial \boldsymbol{f}/\partial t$ together with the orientation of $\tilde{S}$. The time-evolution of these vectors is governed by the differential equations

$$\frac{\mathrm{d}\tilde{\boldsymbol{e}}_i}{\mathrm{d}t} = \boldsymbol{g}_i(\boldsymbol{\Omega}), \quad i = 1, 2, 3, \tag{3.8}$$

where the explicit form of the functions $\boldsymbol{g}_i$ can be obtained by using a specific parameterization for the orientation of $\tilde{S}$, e.g. Eulerian angles. The conceptual separation between the prescribed shape and the unknown orientation is explicit in the functional representations (3.5)–(3.6). Using (3.7), the coupled system (3.4) and (3.8) completely defines the evolution of the swimmer relative to $S$, for any arbitrary body-fixed shape description.

When analysing self-propulsion one is often interested in the efficiency of the swimmer. Efficiency analyses require calculation of the instantaneous power $\mu P$ the swimmer transfers to the fluid:

$$P = \oint_{\partial D} \mathrm{d}A\, \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{v}. \tag{3.9}$$

This integral is decomposed into three terms according to the representation (2.4):

$$P = \oint_{\partial D} \mathrm{d}A\, \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \frac{\mathrm{d}\bar{\boldsymbol{x}}}{\mathrm{d}t} + \oint_{\partial D} \mathrm{d}A\, \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot (\boldsymbol{\Omega} \times \boldsymbol{r}) + \oint_{\partial D} \mathrm{d}A\, \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \frac{\partial \boldsymbol{f}}{\partial t}. \tag{3.10}$$

Since $\mathrm{d}\bar{\boldsymbol{x}}/\mathrm{d}t$ can be taken out of the first integral, the force-free condition (3.1$a$) implies that this integral vanishes. Rearranging the scalar-triple product in the second integral, in conjunction with the torque-free condition (3.1$b$), shows that this integral vanishes too. Upon substitution of the stress decomposition (3.2), the remaining integral is again decomposed into three terms. Making use of the reciprocal relations (3.3) then yields

$$P = \frac{\mathrm{d}\bar{\boldsymbol{x}}}{\mathrm{d}t} \cdot \boldsymbol{F}_D + \boldsymbol{\Omega} \cdot \boldsymbol{T}_D + \oint_{\partial D} \mathrm{d}A\, \boldsymbol{n} \cdot \boldsymbol{\sigma}_D \cdot \frac{\partial \boldsymbol{f}}{\partial t}.$$

† The union of $\tilde{f}_1(t)$, $\tilde{f}_2(t)$, and $\tilde{f}_3(t)$ is equivalent to the complete description of the unlocated shape.

Finally, substitution of (3.4) furnishes the compact quadratic expression

$$P = \boldsymbol{M}_T : \boldsymbol{F}_D\boldsymbol{F}_D + \boldsymbol{M}_R : \boldsymbol{T}_D\boldsymbol{T}_D + 2\boldsymbol{M}_C : \boldsymbol{F}_D\boldsymbol{T}_D + \oint_{\partial D} \mathrm{d}A\, \boldsymbol{n}\cdot\boldsymbol{\sigma}_D\cdot\frac{\partial \boldsymbol{f}}{\partial t}. \tag{3.11}$$

Here, the Chapman–Enskog convention for the double-dot product is employed.

## 4. Symmetric shape deformations

This section addresses swimming strokes which possess two mutually perpendicular planes of symmetry (and, as a particular case, axisymmetric deformations). With no loss of generality, the $\tilde{x}_1$-axis is chosen to lie along the intersection of these two planes. It is clear by symmetry that $\boldsymbol{F}_D$ is directed along the $\tilde{\boldsymbol{e}}_1$-direction and that $\boldsymbol{T}_D$ vanishes. Moreover, for such symmetric shapes it is known (Happel & Brenner 1965) that $\boldsymbol{M}_T$ has the diagonal structure

$$\boldsymbol{M}_T = M_1\tilde{\boldsymbol{e}}_1\tilde{\boldsymbol{e}}_1 + M_2\tilde{\boldsymbol{e}}_2\tilde{\boldsymbol{e}}_2 + M_3\tilde{\boldsymbol{e}}_3\tilde{\boldsymbol{e}}_3, \tag{4.1}$$

while $\boldsymbol{M}_C$ has the form $M_{23}\tilde{\boldsymbol{e}}_2\tilde{\boldsymbol{e}}_3 + M_{32}\tilde{\boldsymbol{e}}_3\tilde{\boldsymbol{e}}_2$ (implying that $\boldsymbol{M}_C\cdot\boldsymbol{F}_D = \boldsymbol{0}$). Thus, (3.4) reveals that at each moment the swimmer translates in the $\tilde{x}_1$-direction and does not rotate. Since the axes $\tilde{x}_i$ $(i = 1, 2, 3)$ retain their original orientation relative to the fluid, they are conveniently assumed parallel to $x_i$. The single-cycle displacement along the $x_1$-axis, $\boldsymbol{d} = d\boldsymbol{e}_1$, is obtained via integration of (3.4) over one period, $0 < t < T$:

$$\boldsymbol{d} = \int_0^T \mathrm{d}t\, \boldsymbol{M}_T[\boldsymbol{f}(t)]\cdot\boldsymbol{F}_D[\boldsymbol{f}(t)].$$

(Since the system $\tilde{S}$ retains it original orientation, the trivial dependence upon its three unit vectors is suppressed, and the set $\{\tilde{f}_i(t)\}$ can be denoted by $\boldsymbol{f}(t)$.) Substitution of (3.5)–(3.7) and use of (4.1) in conjunction with the definition of $\boldsymbol{\Sigma}_T$ provides the following expression for the scalar displacement $d$:

$$d = \int_0^T \mathrm{d}t\, M_1[\boldsymbol{f}(t)] \iint \mathrm{d}s_1\,\mathrm{d}s_2 \left(\frac{\partial \boldsymbol{f}}{\partial s_1}\times\frac{\partial \boldsymbol{f}}{\partial s_2}\right)\cdot\boldsymbol{\sigma}_{T,1}[\boldsymbol{f}(t); s_1, s_2]\cdot\frac{\partial \boldsymbol{f}}{\partial t}. \tag{4.2}$$

The integral (4.2) presents the displacement of the located shape as a functional of the shape evolution of the unlocated shape. Since the flow depends both upon body shape and surface velocity, this functional is nonlinear. Note that $d$ is independent of the fluid viscosity: this is an artifact of the assumption of a prescribed body shape, which is independent of the flow. In principle, $d$ may be evaluated for a prescribed deformation $\boldsymbol{f}(s_1, s_2; t)$ provided the hydrodynamic kernel $\boldsymbol{\sigma}_{T,1}$ is known for the entire sequence of body shapes. (The shape-dependent mobility $M_1$ is automatically furnished by the distribution of $\boldsymbol{\sigma}_{T,1}$ over $\partial D$.) In principle, the stress distribution $\boldsymbol{\sigma}_{T,1}$ can be evaluated numerically (Pozrikidis 1992).

Using (4.2), the 'scallop theorem' is trivial to prove. Indeed, use of the integration variable $\eta = T - t$ yields

$$d = -\int_0^T \mathrm{d}\eta\, M_1[\boldsymbol{f}(T-\eta)]\times\iint \mathrm{d}s_1\,\mathrm{d}s_2\left\{\frac{\partial \boldsymbol{f}}{\partial s_1}(s_1, s_2; T-\eta)\times\frac{\partial \boldsymbol{f}}{\partial s_2}(s_1, s_2; T-\eta)\right\}$$
$$\cdot\boldsymbol{\sigma}_{T,1}[\boldsymbol{f}(T-\eta); s_1, s_2]\cdot\frac{\partial \boldsymbol{f}}{\partial \eta}.$$

For a reversible change of shape, $\boldsymbol{f}(t) = \boldsymbol{f}(T-t)$, the right-hand side is easily seen to be $-d$. Whence $d = 0$.

It is also a simple matter now to display the 'geometric nature' of Stokes swimming. Consider two different strokes: the first, $\boldsymbol{f}(t)$, is executed during the time interval $0 < t < T$, and results in a displacement $d$; the second, $\boldsymbol{f}^*(t)$, is executed over the time interval $0 < t < T^*$ and results in $d^*$. If these strokes are geometrically similar, there exists a monotonic function $\tau(t)$, with $\tau(0) = 0$ and $\tau(T) = T^*$, such that

$$\boldsymbol{f}(t) = \boldsymbol{f}^*(\tau(t)) \tag{4.3}$$

(a simple example is a uniform time-stretch, $\tau(t) = (T^*/T)t$). Substitution of (4.3) in (4.2) and use of the integration variable $\eta = \tau(t)$ yields

$$d = \int_0^{T^*} \mathrm{d}\eta\, M_1[\boldsymbol{f}^*(\eta)] \times \iint \mathrm{d}s_1\, \mathrm{d}s_2 \left\{ \frac{\partial \boldsymbol{f}^*}{\partial s_1}(s_1, s_2; \eta) \times \frac{\partial \boldsymbol{f}^*}{\partial s_2}(s_1, s_2; \eta) \right\} \cdot \boldsymbol{\sigma}_{T,1}[\boldsymbol{f}^*(\eta); s_1, s_2] \cdot \frac{\partial \boldsymbol{f}^*}{\partial \eta}.$$

The right-hand side is clearly $d^*$.

## 5. Concluding remarks

The locomotion of a deformable swimmer in a viscous fluid has been discussed for the general case of arbitrary deformations. The body-fixed swimmer shape is defined using a Lagrangian description, which naturally merges with the flow problem formulation. The equations that describe the swimmer motion are easy to employ in various applications, as they only require the proper delineation of the swimmer intrinsic deformation. This description is rendered 'purely intrinsic' by a consistency relation, which eliminates the translational degree of freedom in the specification of the unlocated body shape. Moreover, the hydrodynamic data which reflect the swimmer resistance to the shape deformation are presented in a convenient way, as body-fixed tensors which can be considered to be known for a given body shape (even if their actual calculation can be computationally demanding).

The underlying idea behind the analysis in this paper is the arbitrary decomposition of the motion into a periodic shape deformation and a rigid-body motion. Naturally, a question arises: is there a physically meaningful way to uniquely decompose the velocity of the swimmer surface into translation, rotation, and deformation? In his analysis of locomotion in a perfect fluid, Miloh (1983) (see also Miloh & Galper 1993) employed such a well-defined decomposition, by attaching the rotating system to the principle axes of the deforming body. In that decomposition, the 'pure deformation' velocity $\partial \boldsymbol{f}/\partial t$ is found to satisfy (in addition to (2.11)) the following kinematic constraints:

$$\oint_{\partial D} \mathrm{d}A\, \boldsymbol{n} \cdot \frac{\partial \boldsymbol{f}}{\partial t} x_i x_j = 0, \qquad i \neq j = 1, 2, 3. \tag{5.1}$$

Unfortunately, this decomposition does not appear suitable for the motion in a viscous fluid: consider the rotation of a sphere about its centre – a motion that trivially satisfies (5.1). In the present context, however, it cannot qualify as 'net distortion' with no rotational component. The incompatibility of Miloh's decomposition with the present Lagrangian formulation, as demonstrated by this rotation example, is attributed to the no-slip boundary condition, which applies in a viscous fluid: in a perfect fluid such rigid-body rotation would not affect the flow field, and would therefore constitute a dead degree of freedom.

The decomposition of the swimmer's shape is more than a mere technicality: it also handles a critical separation between the internal body movements, presumably controlled (via an internal mechanism) by the body, and the external movements, which result from interaction with fluid. This separation can be appreciated by imagining the swimmer to be isolated, unable to interact with any other body (and, specifically, with a surrounding fluid): momentum conservation principles then prevent the swimmer from modifying its centre of mass – regardless of the internal mechanism that affects the shape. Defining the intrinsic shape deformation via a relative description is consistent with this constraint. For Stokes flows, at which inertia is negligible, these momentum arguments may appear superficially confusing. However, the strict connection between Newton's second law and the centre-of-mass position of an isolated system is universal, and is independent of the rheological properties of the fluid in which the body is eventually placed!

## REFERENCES


Avron, J. E., Gat, O. & Kenneth, O. 2004 Optimal swimming at low reynolds numbers. *Phys. Rev. Lett.* **93** (18) 186001.

Becker, L. E., Koehler, S. A. & Stone, H. A. 2003 On self-propulsion of micro-machines at low Reynolds number: Purcell's three-link swimmer. *J. Fluid Mech.* **490**, 15–35.

Blake, J. R. 1971 Self propulsion due to oscillations of a cylinder at low Reynolds number. *Bull. Austral. Math. Soc.* **3**, 255–264.

Brennen, C. & Winet, H. 1977 Fluid-mechanics of propulsion by cilia and flagella. *Annu. Rev. Fluid Mech.* **9**, 339–398.

Brenner, H. 1964 The Stokes resistance of an arbitrary particle — IV. Arbitrary fields of flow. *Chem. Engng Sci.* **19**, 703–727.

Childress, S. 1981 *Mechanics of Swimming and Flying*. Cambridge University Press.

Happel, J. & Brenner, H. 1965 *Low Reynolds Number Hydrodynamics*. Prentice-Hall.

Kim, S. & Karrila, S. J. 1991 *Microhydrodynamics*. Butterworth-Heinemann.

Lighthill, J. 1952 On the squirming motion of nearly spherical deformable bodies through liquids at very small Reynolds numbers. *Commun. Pure Appl. Maths* **5**, 109–118.

Lighthill, M. J. 1975 *Mathematical Biofluiddynamics*. SIAM.

Magar, V. & Pedley, T. J. 2005 Average nutrient uptake by a self-propelled unsteady squirmer. *J. Fluid Mech.* **539**, 93–112.

Miloh, T. 1983 Optimal self-propulsion of a deformable prolate spheroid. *J. Ship Res.* **27** (2), 121–130.

Miloh, T. 1991 Hydrodynamic self-propulsion of deformable bodies and oscillating bubbles. In *Mathematical Approaches in Hydrodynamics* (ed. T. Miloh), pp. 21–36. SIAM.

Miloh, T. & Galper, A. 1993 Self-propulsion of general deformable shapes in a perfect fluid. *Proc. R. Soc. Lond.* A **442**, 273–299.

Pozrikidis, C. 1992 *Boundary Integral and Singularity Methods for Linearized Viscous Flow*. Cambridge University Press.

Purcell, E. M. 1977 Life at low reynolds-number. *Am. J. Phys.* **45** (1), 3–11.

Shapere, A. & Wilczek, F. 1989 Geometry of self-propulsion at low Reynolds number. *J. Fluid Mech.* **198**, 557–585.

Stone, H. A. & Samuel, A. D. T. 1996 Propulsion of microorganisms by surface distortions. *Phys. Rev. Lett.* **77**, 4102–4104.

Taylor, G. I. 1951 Analysis of the swimming of microscopic animals. *Proc. R. Soc. Lond.* A **209**, 447–461.